On the load–displacement relations used in the analysis of nanoindentation data

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Nanoindentation experiments have become a commonly used technique to investigate the mechanical properties of thin films and small volumes of materials. A power-law load–displacement (*P*–*h*) relation is generally used in the interpretation of naonindentation experimental data [1]. The power-law assumption has been proved to be true for flat-ended, spherical, and conical indenters [2]. However, there is no study on whether this assumption is valid for other smooth indenters. In this paper, a general polynomial indenter is considered. For a power-law indenter, its load–displacement follows a power-law relationship. It becomes complicated for a non-power-law polynomial indenter. An algebraic equation has to be solved in order to obtain the load–displacement relationship, which is not a power-law formula. This leads to the conclusion that the load–displacement for a smooth indenter does not necessarily follow a power-law relation.

We consider a rigid smooth frictionless axisymmetric indenter with its axis of revolution as the z-axis indenting normally into the plane $z = 0$ of an elastic half-space $z \geq 0$. The problem is considered in the linear theory of elasticity and the half-space is assumed to be isotropic and homogeneous. The contact region between the indenter and the half-space is simply connected.

The following equations give the relevant displacement and stresses for the half-space. The vertical component of the displacement is denoted by u_z, and the stress components have two subscripts corresponding to the appropriate coordinates. E and ν are Young's modulus and Poisson's ratio of the half-space, respectively.

As Fig. 1 shows, the boundary conditions for the half-space at $z = 0$ are

$$\tau_{zr} = \tau_{z\theta} = 0 \quad (0 \leq r < \infty) \tag{1}$$

$$\sigma_{zz} = 0 \quad (r > a) \tag{2}$$

$$u_z = h + \sum_{\alpha=\alpha_1}^{\alpha_n} a_\alpha r^\alpha \quad (0 \leq r \leq a) \tag{3}$$

where α is a positive real number. The second term on the right-hand side of Equation 3 describes the indenter shape. If it is zero, Equation 3 refers to a flat-ended indenter.

If an indenter is not flat-ended, its radius of contact area, a, and its depth of penetration, h, are related by the following equation [3]:

$$\frac{2}{\sqrt{\pi}}h + \sum_{\alpha=\alpha_1}^{\alpha_n}(1+\alpha)a_\alpha \frac{\Gamma((2+\alpha)/2)}{\Gamma((3+\alpha)/2)}a^\alpha = 0. \tag{4}$$

The total vertical load, P, which causes the displacement, h, is

$$P = \sqrt{\pi}\frac{E}{1-\nu^2}\left[\frac{2}{\sqrt{\pi}}ah + \sum_{\alpha=\alpha_1}^{\alpha_n} a_\alpha \frac{\Gamma((2+\alpha)/2)}{\Gamma((3+\alpha)/2)}a^{1+\alpha}\right]. \tag{5}$$

From Equation 4, Equation 5 can be rewritten as the following format:

$$P = -\sqrt{\pi}\frac{E}{1-\nu^2}\sum_{\alpha=\alpha_1}^{\alpha_n}\left[a_\alpha \alpha \frac{\Gamma((2+\alpha)/2)}{\Gamma((3+\alpha)/2)}a^{1+\alpha}\right]. \tag{6}$$

As special cases of the general solution, power-law, polynomial and smooth indenters are considered in the following discussions.

The vertical displacement due to the penetration of a power-law indenter is

$$u_z = h + a_\alpha r^\alpha \quad (0 \leq r \leq a) \tag{7}$$

where α is a positive real number and the indenter shape is described by a single term. When $\alpha = 1$, Equation 7 refers to a conical indenter; and, when $\alpha = 2$, it is for a parabolic indenter.

From the contact radius and the indentation depth relationship, we have

$$a = \left[-\frac{2}{\sqrt{\pi}}\frac{1}{1+\alpha}\frac{\Gamma((3+\alpha)/2)}{\Gamma((2+\alpha)/2)}\frac{1}{a_\alpha}\right]^{\frac{1}{\alpha}} h^{\frac{1}{\alpha}}. \tag{8}$$

Putting Equation 8 into the load–displacement relation, we have

$$P = 2\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{\alpha}}\frac{E}{1-\nu^2}\frac{\alpha}{(1+\alpha)^{\frac{1+\alpha}{\alpha}}}\left[\frac{\Gamma((3+\alpha)/2)}{\Gamma((2+\alpha)/2)}\right]^{\frac{1}{\alpha}} \times \frac{1}{(-a_\alpha)^{\frac{1}{\alpha}}}h^{\frac{1+\alpha}{\alpha}}. \tag{9}$$

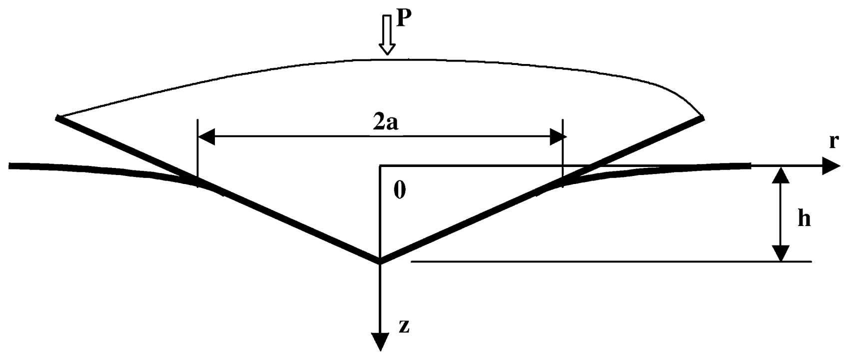

Figure 1 Normal indentation of an elastic half-space.

Equation 9 shows that for a power-law indenter, its load–displacement follows a power-law relationship.

The vertical displacement due to the penetration of a general polynomial indenter is

$$u_z = h + \sum_{i=1}^{n} a_i r^i \quad (0 \leq r \leq a) \tag{10}$$

where at least two of the a_i $(i = 1, 2, \ldots, n)$ are not zero. This restriction excludes the power-law indenter cases.

From Equation 4, we have

$$\frac{2}{\sqrt{\pi}} h + \sum_{i=1}^{n} (1+i)\, a_i \,\frac{\Gamma((2+i)/2)}{\Gamma((3+i)/2)}\, a^i = 0. \tag{11}$$

From Equation 11, we can express the contact area by the indentation depth as a function $a = a(h)$. It will be difficult if Equation 11 is an algebraic equation of high order. If we put the function $a = a(h)$ into Equation 5, we would obtain the load–displacement relation. For a non-power-law polynomial indenter described by Equation 10, it is obvious that its load–displacement curve does not follow a power-law relationship.

A smooth indenter can be described as a solid of revolution of a smooth function, and this function can be expanded mathematically as a polynomial series, i.e., Maclaurin series. Because the load–displacement (P–h) curve for a polynomial indenter does not always follow a power-law relation, the P–h relation for a smooth indenter is not necessarily a power-law formula.

References

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